can show substantial gains in the sample length required to achieve a given variability. Extensive vertification of our scheme has been carried out on simulated data. In one practical but rather limited example the technique was shown to reduce the variability of the estimates by some 60% as predicted by our theory for the parameters arising in that case. We feel that Bell's dismissal of this approach is quite unfounded.

We are well aware that Eqs. (1) and (2) are not computationally very efficient, and went to some pains in Refs. 2 and 3 to discuss faster implementations of these schemes. Indeed, we have considered the slotted correlation approach and have shown by numerical experiment that the variability is similar to that predicted for Eq. (1). In fact, this has been shown to be a practical and fairly straightforward way of getting spectra from randomly acquired data, and have been involved in its use in Refs. 6 and 7. We also suggest various ways of using a look-up table or FFT method in the direct transform. It is not helpful at all to compare Eq. (2) with a slotted correlation approach insofar as speed is concerned. If speed is of extreme significance then our proposals in Ref. 8 on a rapid estimator should be examined.

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# Reply by Author to M. Gaster and J. B. Roberts

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THE comments made by Gaster and Roberts concerning Ref. 1 provide a welcome opportunity to clarify some of the points brought out in the paper. First of all, the unbiased estimator for the power spectrum based on the periodogram is

$$\hat{S}(\omega)=\frac{T}{N^2}\left\{\left|\sum_k X_k e^{i\omega t_k}\right|^2-\sum_k X_k^2\right\} \tag{1}$$

This equation should be compared with Eq. (7) of Ref. 1. In the reference it should have been clearly stated that Eq. (7) yields a biased estimate and that the approach used in the paper was an attempt to circumvent the effects of that bias. However, as Gaster and Roberts point out and as a comparison of the results presented in their Comment with the results of Fig. 2 in the paper indicate, this perhaps ingenious approach is a poor substitute for the direct approach of Ref. 2. This, then, explains the discrepancy in the variability predicted from the rigorous analysis in Ref. 2 and that observed in the paper.

The results in the paper were obtained by taking the logarithm of the magnitude of the spectral estimate and multiplying by ten. This was done to highlight the variability of the spectrum. Of course Eq. (1) can produce negative values, but it was felt that the magnitude of the spectral estimate was most germane to the study of variability. Using Eq. (1), the peak at the expected frequency occurs 20 DB above the estimators at the "background" frequencies as shown by the results presented in the comments by Gaster and Roberts.

As Gaster and Roberts also demonstrate, the variability can be reduced by a suitable choice of sampling rate when using Eq. (1). In cases where this approach is unfeasible, the author has found in recent research that the relatively simple technique described in Ref. 3 can be used to decrease the variability for certain spectra.

The author shares the commentors' opinion that the subject matter of the paper is very relevant to data analysis with the laser velocimeter and hopes that further research into this important field is forthcoming.

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# Comment on "Generalized Coordinate Forms of Governing Fluid Equations and Associated Geometrically Induced Errors"

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IN a recent paper,[1] Hindman presented an analysis regarding geometrically induced errors arising from conservation laws written in a generalized coordinate system. Specifically, the observation is made that if a numerical integration algorithm is consistently formulated, it should be able to integrate a constant uniform flow solution without producing any changes. An analysis is performed by Hindman on two-dimensional chain rule, weak, and strong conservation law forms (CRCLF, WCLF, and SCLF, respec-

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tively) using MacCormack's explicit unsplit predictor-corrector integration scheme.

The result obtained is that the uniform flow reproduction test is automatically satisfied when the CRCLF equation is used. Using conventional forward ($\Delta$) and backward ($\nabla$) difference operators for the predictor and corrector, respectively, it is found that the test is satisfied by the WCLF equation only if the metric derivatives are computed numerically with the same difference operator that is used to evaluate the associated flux derivative. That is,

$$(\xi_t)_\xi \simeq \frac{\Delta_i(\xi_t)}{\Delta\xi}, \quad (\xi_x)_\xi \simeq \frac{\Delta_i(\xi_x)}{\Delta\xi}, \text{ etc.} \tag{1}$$

for the predictor step, and

$$(\xi_t)_\xi^{\overline{n+1}} \simeq \frac{\nabla_i(\overline{\xi_t^{n+1}})}{\Delta\xi}, \quad (\xi_x)^{\overline{n+1}} \simeq \frac{\nabla_i(\overline{\xi_x^{n+1}})}{\Delta\xi}, \text{ etc.} \tag{2}$$

for the corrector step. Hindman notes that no constraint is placed on how the metrics $\xi_t$, $\xi_x$, etc., are to be evaluated but only how their derivatives must be evaluated. This is in distinction with the SCLF equations which Hindman shows that the metrics themselves are to be computed numerically with the same difference operator that is used to evaluate the associated flux derivative. That is,

$$x_\xi \simeq \frac{\Delta_i(x)}{\Delta\xi}, \quad y_\xi \simeq \frac{\Delta_i(y)}{\Delta\xi}, \text{ etc.} \tag{3}$$

for the predictor, and

$$x_\xi^{\overline{n+1}} \simeq \frac{\nabla_i(x)^{\overline{n+1}}}{\Delta\xi}, \quad y_\xi^{\overline{n+1}} \simeq \frac{\nabla_i(y)^{\overline{n+1}}}{\Delta\xi}, \text{ etc.} \tag{4}$$

for the corrector. In addition, the geometric conservation law (GCL) originally given by Thomas and Lombard[2] must also be integrated using MacCormack's scheme.

The implication by Hindman is that all results (or restrictions to satisfy the test) easily extend to three dimensions. This is not the case for the SCLF equation. If the results are simply extended to three dimensions, to satisfy the test we need to impose the conditions

$$\frac{\Delta_i(\xi_x/J)}{\Delta\xi} + \frac{\Delta_j(\eta_x/J)}{\Delta\eta} + \frac{\Delta_k(\zeta_x/J)}{\Delta\zeta} = 0, \text{ etc.} \tag{5}$$

for the predictor, and

$$\frac{\nabla_i(\xi_x/J)^{\overline{n+1}}}{\Delta\xi} + \frac{\nabla_j(\eta_x/J)^{\overline{n+1}}}{\Delta\eta} + \frac{\nabla_k(\zeta_x/J)^{\overline{n+1}}}{\Delta\zeta} = 0, \text{ etc.} \tag{6}$$

for the corrector. Extending results obtained by Hindman, Eq. (5), for example, would be rewritten (using transformation identities) as

$$\frac{\Delta_i(y_\eta z_\zeta - y_\zeta z_\eta)}{\Delta\xi} + \frac{\Delta_j(y_\zeta z_\xi - y_\xi z_\zeta)}{\Delta\eta} + \frac{\Delta_k(y_\xi z_\eta - y_\eta z_\xi)}{\Delta\zeta} = 0 \tag{7}$$

Now evaluating the metrics using forward difference operators it can be shown that it does not yield an identity, contrary to Hindman's two-dimensional results.

To obtain an identity, Eq. (7) should be written in the following form

$$\frac{\Delta_i[(y_\eta z)_\zeta - (y_\zeta z)_\eta]}{\Delta\xi} + \frac{\Delta_j[(y_\zeta z)_\xi - (y_\xi z)_\zeta]}{\Delta\eta} + \frac{\Delta_k[(y_\xi z)_\eta - (y_\eta z)_\xi]}{\Delta\zeta} = 0 \tag{8}$$

and similarly for the other equations. Now it is a simple matter to show that if we apply the forward difference operator to the above equation, i.e.,

$$(y_\eta z)_\zeta \simeq \frac{\Delta_k(y_\eta z)}{\Delta\zeta}, \quad (y_\zeta z)_\eta \simeq \frac{\Delta_j(y_\zeta z)}{\Delta\eta}, \text{ etc.} \tag{9}$$

and since $\Delta_i(\Delta_j A) = \Delta_j(\Delta_i A)$, then Eq. (8) is identically satisfied. The same result is obtained for all other predictor and corrector constraint equations since $\nabla_i(\nabla_j A) = \nabla_j(\nabla_i A)$ also holds. It is now noted that, contrary to Hindman's SCLF two-dimensional results, no constraint is placed on how the metrics are to be evaluated. Of course the GCL equation must still be integrated using MacCormack's scheme when the grids move.

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# Reply by Author to S. Paolucci

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Mr. Paolucci's work is essentially correct. However, Thomas and Lombard[1] recognized in 1978 that, for three-dimensional geometry, the geometric consistency required for the strong conservation law form of the governing equations is accomplished automatically by writing the conservative form of the metrics. For example,

$$\xi_x/J = \hat{i}\cdot\bar{r}_\eta \times \bar{r}_\zeta \quad (\bar{r} \text{ is the position vector})$$

is written conservatively as

$$\xi_x/J = (y_\eta z)_\zeta - (y_\zeta z)_\eta$$

Now, subsequent differencing of this relation with the same difference relations as used for the flux quantities will always yield exact geometric error cancellation provided the difference operators are linear. Therefore, Paolucci's Eq. (8)

$$\frac{\Delta_i[(y_\eta z)_\zeta - (y_\zeta z)_\eta]}{\Delta\xi} + \frac{\Delta_j[(y_\zeta z)_\xi - (y_\xi z)_\zeta]}{\Delta\eta} + \frac{\Delta_k[(y_\xi z)_\eta - (y_\eta z)_\xi]}{\Delta\zeta} = 0$$

is not new.

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